4.4 – Linear Independence

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a **linearly independent set** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be **linearly dependent**. If S has only one vector, we will agree that it is linearly independent if and only if that vector is nonzero.

Theorem 4.4.1 A nonempty set $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation $k_1\,\mathbf{v}_1 + k_2\,\mathbf{v}_2 + \dots + k_r\,\mathbf{v}_r = \mathbf{0}$ are $k_1 = 0,\ k_2 = 0,\ \dots,\ k_r = 0$.

#3 In each part, determine whether the vectors are linearly independent or are linearly dependent in R^4.

a. $(3,\ 8,\ 7,\ -3), (1,\ 5,\ 3,\ -1), (2,\ -1,\ 2,\ 6), (4,\ 2,\ 6,\ 4)$

b. $(3,\ 0,\ -3,\ 6), (0,\ 2,\ 3,\ 1), (0,\ -2,\ -2,\ 0), (-2,\ 1,\ 2,\ 1)$

#5 In each part, determine whether the matrices are linearly independent or dependent.

a. $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ in M_{22}

b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

#9

a. Show that the three vectors $\mathbf{v}_1 = (0,\ 3,\ 1,\ -1)$, $\mathbf{v}_2 = (6,\ 0,\ 5,\ 1)$, $\mathbf{v}_3 = (4,\ -7,\ 1,\ 3)$ form a linearly dependent set in R^4.

b. Express each vector in part (a) as a linear combination of the other two.

#14 In each part, let $T_A : R^3 \to R^3$ be multiplication by A and let $\mathbf{u}_1 = (1,\ 0,\ 0)$, $\mathbf{u}_2 = (2,\ -1,\ 1)$, and $\mathbf{u}_3 = (0,\ 1,\ 1)$. Determine whether the set $\{T_A(\mathbf{u}_1),\ T_A(\mathbf{u}_2),\ T_A(\mathbf{u}_3)\}$ is linearly independent in R^3.

a. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

Theorem 4.4.2
a) A set with finitely many vectors that contains **0** is linearly dependent.
b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Theorem 4.4.3 Let $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

Definition: If $\mathbf{f}_1 = f_1(x)$, $\mathbf{f}_2 = f_2(x)$, $\dots$, $\mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the **Wronskian** of $f_1, f_2, \dots, f_n$.

Theorem 4.4.4 If the functions $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ have $n-1$ continuous derivatives on the interval $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then these functions form a linearly independent set of vectors in $C^{(n-1)}(-\infty, \infty)$.

#19 Use the Wronskian to show that the following sets of vectors are linearly independent.

a. $1, x, e^x$

b. $1, x, x^2$